Available online at www.sciencedirect.com
SCIENCE DIRECT®

Fuzzy Sets and Systems 157 (2006) 1970–1982

FUZZY sets and systems
www.elsevier.com/locate/fss

On initial and final fuzzy uniform structures, Part II

Fatma Bayoumi*

Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt

Received 11 June 2005; received in revised form 16 February 2006; accepted 8 March 2006
Available online 30 March 2006

Abstract

This paper is the second part and continuation of a paper for the author published in 2003 and investigating initial fuzzy uniform structures. The final fuzzy uniform structures and the final global fuzzy neighborhood structures, for the notions of fuzzy uniform structure and of global fuzzy neighborhood structure introduced by the author and others in 1998 in two separate papers, are characterized. This paper also shows that the expected relations between the final fuzzy uniform structures and the final fuzzy topologies and the final global fuzzy neighborhood structures are indeed true.

MSC: primary 54; secondary 54E

Keywords: Fuzzy uniform structures; Initial and final fuzzy uniform structures; Global fuzzy neighborhood structures; Initial and final global fuzzy neighborhood structures; Global homogeneous fuzzy neighborhood structures; Initial and final fuzzy topologies; Initial and final lifts

1. Introduction

This paper is the second part and continuation of [2] investigating initial and final fuzzy uniform structures. Whereas [2] dealt with initial structures, this paper deals with final structures.

For the notion of fuzzy uniform structure introduced by the author and others in [7], we showed in [2] that the category FUN of fuzzy uniform spaces is topological over SET with respect to the expected forgetful functor. Hence, the final lifts and thus the final fuzzy uniform structures exist [1,19]. In this paper, we characterize these final fuzzy uniform structures and we show that they provide final lifts. We also show that the fuzzy topology associated with the final fuzzy uniform structure of a family $(\mathcal{U}_i)_{i\in I}$ of fuzzy uniform structures $\mathcal{U}_i$ coincides with the final fuzzy topology of the family $(\tau_{\mathcal{U}_i})_{i\in I}$ of fuzzy topologies $\tau_{\mathcal{U}_i}$ associated with $\mathcal{U}_i$.

We also characterize here the final global fuzzy neighborhood structures, that is, the global fuzzy neighborhood structures which provide final lifts in the category FNS of global fuzzy neighborhood spaces [6]. It is shown that the global fuzzy neighborhood structures associated with the final fuzzy uniform structure of a family $(\mathcal{U}_i)_{i\in I}$ of fuzzy uniform structures $\mathcal{U}_i$ coincides with the final global fuzzy neighborhood structure of the family $(h_{\mathcal{U}_i})_{i\in I}$ of global fuzzy neighborhood structures $h_{\mathcal{U}_i}$ associated with $\mathcal{U}_i$. Moreover, the relation between the final global fuzzy neighborhood structure and the final fuzzy topology is fulfilled, that is, the fuzzy topology associated with the final global fuzzy neighborhood structure of a family $(h_i)_{i\in I}$ of global fuzzy neighborhood structures h_i coincides with the final fuzzy

* Tel./fax: +2013220919.
E-mail address: fatma_bayoumi@hotmail.com.

0165-0114/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2006.03.012

topology of the family $(\tau_{h_i})_{i\in I}$ of fuzzy topologies τ_{h_i} associated with h_i. The expected relation between the initial global fuzzy neighborhood structure and the initial fuzzy topology also is verified.

This paper and its predecessor comprise a generalization by filters of the entourage approach of Lowen [18] and Höhle [10] for fuzzy sets, which goes back to the entourage approach of Weil [23]. The reader should be aware that there are other approaches, including the uniform covering approach of Tukey [22] and Isbell [15] and the closely related uniform operator approach articulated in Kotzé [16,12], but which was first given in the fuzzy setting by Hutton [14]. With the exception of the generalization of Hutton in Rodabaugh [20,21] and the modification of Hutton in Zhang [24], these three approaches have been unified in Gutiérrez García et al. [9,21] using the filter approach of Höhle and Šostak [12,13] and Höhle [11,12]. It remains for future work to clarify the relationship of the filter approach of this paper with that of Gutiérrez García et al. [9,21].

In the following lines we recall some definitions and results, which we need in this paper, related to fuzzy filters, fuzzy uniform structures and global fuzzy neighborhood structures.

1.1. Fuzzy filters

Let L be completely distributive complete lattice with different least and greatest elements 0 and 1, respectively. L^X denotes the set of all fuzzy subsets of a set X. By a *fuzzy filter* $\mathcal{M}$ on X is meant a mapping $\mathcal{M} : L^X \to L$ [5] such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\bar{\alpha}) \leqslant \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\bar{1}) = 1$.
(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

A fuzzy filter $\mathcal{M}$ on X is called *homogeneous* [4] if $\mathcal{M}(\bar{\alpha}) = \alpha$ for all $\alpha \in L$. Denote by $\mathcal{F}_L X$ and $\mathsf{F}_L X$ the sets of all fuzzy filters and of all homogeneous fuzzy filters on X, respectively. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on X, $\mathcal{M}$ is called *finer* than $\mathcal{N}$ or $\mathcal{N}$ is called *coarser* than $\mathcal{M}$, denoted by $\mathcal{M} \leqslant \mathcal{N}$, provided $\mathcal{M}(f) \geqslant \mathcal{N}(f)$ holds for all $f \in L^X$.

The *image* of a fuzzy filter $\mathcal{M}$ on X with respect to a mapping $f : X \to Y$ is the fuzzy filter $\mathcal{F}_L f(\mathcal{M})$ on Y defined by $\mathcal{F}_L f(\mathcal{M})(g) = \mathcal{M}(g \circ f)$ for all $g \in L^Y$ [5]. If $\mathcal{N}$ is a fuzzy filter on Y, then the *preimage* $\mathcal{F}_L^- f(\mathcal{N})$ of $\mathcal{N}$ with respect to f does not exist, in general, as a fuzzy filter. If $\mathcal{F}_L^- f(\mathcal{N})$ exists, then it is the coarsest fuzzy filter $\mathcal{M}$ on X for which $\mathcal{F}_L f(\mathcal{M}) \leqslant \mathcal{N}$ holds, that is, for all $g \in L^X$ we have [5]:

$$\mathcal{F}_L^- f(\mathcal{N})(g) = \bigvee_{h \circ f \leqslant g} \mathcal{N}(h).$$

The image and preimage operators $\mathcal{F}_L f : L^{(L^X)} \to L^{(L^Y)}$ and $\mathcal{F}_L^- f : L^{(L^Y)} \to L^{(L^X)}$ are combinations of the Zadeh image and preimage operators $f_L^{\to} : L^X \to L^Y$ and $f_L^{\leftarrow} : L^Y \to L^X$ defined by

$$f_L^{\to}(g)(y) = \bigvee\{g(x) \mid x \in f^{\leftarrow}(\{y\})\} \quad \text{and} \quad f_L^{\leftarrow}(h) = h \circ f$$

for all $g \in L^X$ and $h \in L^Y$, where $f^{\leftarrow} : \wp(X) \leftarrow \wp(Y)$ is the traditional preimage operator. We have

$$\mathcal{F}_L f = [f_L^{\leftarrow}]_L^{\leftarrow} : L^{(L^X)} \to L^{(L^Y)},$$

and

$$\mathcal{F}_L^- f = [f_L^{\to}]_L^{\leftarrow} : L^{(L^Y)} \to L^{(L^X)}.$$

The properties of $f_L^{\leftarrow}$ guarantee that the image of a fuzzy filter is a fuzzy filter and since the Zadeh image operator $f_L^{\to}$ need not preserve meets, then the preimage of a fuzzy filter need not be a fuzzy filter. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on X and Y, respectively, then

$$\mathcal{M} \leqslant \mathcal{F}_L^- f(\mathcal{F}_L f(\mathcal{M})) \quad \text{and} \quad \mathcal{F}_L f(\mathcal{F}_L^- f(\mathcal{N})) \leqslant \mathcal{N}$$

even if the preimage $\mathcal{F}_L^- f(\mathcal{N})$ is not a fuzzy filter, but only an isotone mapping.

We also have the following result.

Proposition 1.1 (*Gähler et al. [6]*). *Let* $f : X \to Y$ and $g : Y \to Z$ *be mappings and* $\mathcal{N}$ *a fuzzy filter on Z. Then*

$$\mathcal{F}_L^{-}(g \circ f)(\mathcal{N}) \leqslant \mathcal{F}_L^{-} f(\mathcal{F}_L^{-} g(\mathcal{N}))$$

holds.

1.2. Fuzzy uniform structures

For all $x, y \in X$, the mapping $(x, y)^{\bullet} : L^{X\times X} \to L$, defined by $(x, y)^{\bullet}(u) = u(x, y)$ for all $u \in L^{X\times X}$, is a homogeneous fuzzy filter on $X \times X$. The *inverse* of a fuzzy filter $\mathcal{U}$ on $X \times X$ is the fuzzy filter $\mathcal{U}^{-1}$ on $X \times X$ defined by $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ for all $u \in L^{X\times X}$, where u^{-1} is the inverse relation of u defined by $u^{-1}(x, y) = u(y, x)$ for all $x, y \in X$. For any two fuzzy filters $\mathcal{U}$ and $\mathcal{V}$ on $X \times X$ such that $(x, y)^{\bullet} \leqslant \mathcal{U}$ and $(y, z)^{\bullet} \leqslant \mathcal{V}$ hold for some $x, y, z \in X$, the *composition* of $\mathcal{U}$ and $\mathcal{V}$ is the fuzzy filter $\mathcal{V} \diamond \mathcal{U}$ on $X \times X$ defined by

$$(\mathcal{V} \diamond \mathcal{U})(w) = \bigvee_{v\diamond u \leqslant w} (\mathcal{U}(u) \wedge \mathcal{V}(v))$$

for all $w \in L^{X\times X}$, where $v \diamond u$ is the composition of the fuzzy relations u and v on X defined as the fuzzy relation on X by

$$(v \diamond u)(x, y) = \bigvee_{z\in X} (u(x, z) \wedge v(z, y))$$

for all $x, y \in X$ [7]. Note that in the whole paper we are using the notation "$\diamond$" for the composition of fuzzy relations and of fuzzy filters and using the notation "$\circ$" for the composition of mappings as usual.

A *fuzzy uniform structure* $\mathcal{U}$ on a set X [7] is a fuzzy filter on $X \times X$ such that the following conditions are fulfilled:

(U1) $(x, x)^{\bullet} \leqslant \mathcal{U}$ for all $x \in X$.
(U2) $\mathcal{U} = \mathcal{U}^{-1}$.
(U3) $\mathcal{U} \diamond \mathcal{U} \leqslant \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a *fuzzy uniform space*. The mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ between fuzzy uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is called *fuzzy uniformly continuous* if the following holds:

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \leqslant \mathcal{V}.$$

For a fuzzy filter $\mathcal{U}$ on $X \times X$ such that $(x, x)^{\bullet} \leqslant \mathcal{U}$ holds for all $x \in X$, and a fuzzy filter $\mathcal{M}$ on X, the mapping $\mathcal{U}[\mathcal{M}] : L^X \to L$, defined by

$$\mathcal{U}[\mathcal{M}](f) = \bigvee_{u[g]\leqslant f} (\mathcal{U}(u) \wedge \mathcal{M}(g))$$

for all $f \in L^X$, is a fuzzy filter on X, called the *image* of $\mathcal{M}$ with respect to $\mathcal{U}$, where $u[g]$ is the fuzzy subset of X defined by $u[g](x) = \bigvee_{y\in X}(g(y) \wedge u(y, x))$. To each fuzzy uniform structure $\mathcal{U}$ on X is associated a stratified fuzzy topology $\tau_{\mathcal{U}}$ given by

$$\tau_{\mathcal{U}} = \{f \in L^X \mid f(x) = \mathcal{U}[\dot{x}](f) \quad \text{for all } x \in X\},$$

where $\dot{x}$ is a homogeneous fuzzy filter on X defined by $\dot{x}(f) = f(x)$ for each $f \in L^X$ [7]. The fuzzy topology here is in sense of [3,8].

Proposition 1.2 (*Gähler et al. [7]*). *Let* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *be a fuzzy uniformly continuous mapping between fuzzy uniform spaces. Then the mapping* $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ *between the associated fuzzy topological spaces is fuzzy continuous*.

1.3. Global fuzzy neighborhood structures

Since for each homogeneous fuzzy filter $\mathcal{M}$ the image $\mathcal{F}_L f(\mathcal{M})$ also is homogeneous, then $\mathcal{F}_L f$ has the domain–codomain restriction to a mapping of $\mathsf{F}_L X$ into $\mathsf{F}_L Y$, denoted by $\mathsf{F}_L f$.

The mapping $\mathcal{F}_L : \mathsf{SET} \to \mathsf{SET}$ which assigns to each set X the set $\mathcal{F}_L X$ and to each mapping $f : X \to Y$ the mapping $\mathcal{F}_L f : \mathcal{F}_L X \to \mathcal{F}_L Y$ is a covariant functor, called the *fuzzy filter functor*. The subfunctor $\mathsf{F}_L : \mathsf{SET} \to \mathsf{SET}$ of $\mathcal{F}_L$ which assigns to each set X the set $\mathsf{F}_L X$ and to each mapping $f : X \to Y$ the mapping $\mathsf{F}_L f$ is called the *homogeneous fuzzy filter functor*.

$\eta = (\eta_X)_{X \in \mathrm{Ob}(\mathsf{SET})} : \mathrm{id} \to \mathcal{F}_L$ and $\mu = (\mu_X)_{X \in \mathrm{Ob}(\mathsf{SET})} : \mathcal{F}_L \circ \mathcal{F}_L \to \mathcal{F}_L$ are natural transformations, where id means the identity set functor, and $\eta_X : X \to \mathcal{F}_L X$ and $\mu_X : \mathcal{F}_L \mathcal{F}_L X \to \mathcal{F}_L X$ are the mappings defined by $\eta_X(x) = \dot{x}$ and $\mu_X(\mathcal{L}) = \mathcal{L} \circ e_X$ for all $x \in X$ and all $\mathcal{L} \in \mathcal{F}_L \mathcal{F}_L X$, and $e_X : L^X \to L^{\mathcal{F}_L X}$ is the mapping given by $e_X(f)(\mathcal{M}) = \mathcal{M}(f)$ for all $f \in L^X$ and all $\mathcal{M} \in \mathcal{F}_L X$. $(\mathcal{F}_L, \eta, \mu)$ is a monad in the categorical sense, called the *fuzzy filter monad* [5].

Taking the subfunctor F_L instead of $\mathcal{F}_L$, we analogously get natural transformations η' and μ' such that $(\mathsf{F}_L, \eta', \mu')$ is a submonad of $(\mathcal{F}_L, \eta, \mu)$, called the *homogeneous fuzzy filter monad* [4].

A *global fuzzy neighborhood structure* on a set X [6] is defined as a mapping $h : \mathcal{F}_L X \to \mathcal{F}_L X$ such that the following conditions are fulfilled:

(N1) $\mathcal{M} \leqslant h(\mathcal{M})$ holds for all $\mathcal{M} \in \mathcal{F}_L X$.
(N2) $h(\mathcal{L} \vee \mathcal{M}) = h(\mathcal{L}) \vee h(\mathcal{M})$ for all $\mathcal{L}, \mathcal{M} \in \mathcal{F}_L X$.
(N3) $h \circ h = h$ (h is idempotent).
(N4) $\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X \leqslant h$ holds.

A *global homogeneous fuzzy neighborhood structure* is defined analogously as the global fuzzy neighborhood structure, however, by using the homogeneous fuzzy filter monad $(\mathsf{F}_L, \eta', \mu')$ instead of the fuzzy filter monad $(\mathcal{F}_L, \eta, \mu)$.

For each fuzzy uniform structure $\mathcal{U}$ on a set X is associated a global homogeneous fuzzy neighborhood structure $h_{\mathcal{U}}$ on X defined by

$$h_{\mathcal{U}}(\mathcal{M}) = \mathcal{U}[\mathcal{M}]$$

for all $\mathcal{M} \in \mathsf{F}_L X$ [7]. Moreover, each global fuzzy neighborhood structure h on X is associated a fuzzy topology τ_h on X [6] defined by

$$\tau_h = \{f \in L^X \mid f(x) = h(\dot{x})(f) \text{ for all } x \in X\}.$$

If h and k are global fuzzy neighborhood structures on a set X, then h is called finer than k, denoted by $h \leqslant k$, if the fuzzy filter $h(\mathcal{M})$ is finer than $k(\mathcal{M})$ for all $\mathcal{M} \in \mathcal{F}_L X$.

A set X equipped with a global fuzzy neighborhood structure h on X is called a *global fuzzy neighborhood space*. A mapping $f : (X, h) \to (Y, k)$ between global fuzzy neighborhood spaces is called (h, k)*-continuous* [6] provided

$$\mathcal{F}_L f \circ h \leqslant k \circ \mathcal{F}_L f$$

holds.

Proposition 1.3 (*Gähler et al. [7]*). *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a fuzzy uniformly continuous mapping between fuzzy uniform spaces. Then the mapping $f : (X, h_{\mathcal{U}}) \to (Y, h_{\mathcal{U}})$ between the associated global homogeneous fuzzy neighborhood spaces is $(h_{\mathcal{U}}, h_{\mathcal{V}})$-continuous.*

2. Final fuzzy uniform structures

The final structures of the notion of fuzzy uniform structure presented in [7] will be characterized in this section. In case of one mapping we have the following result.

Proposition 2.1. *Let $(X, \mathcal{U})$ be a fuzzy uniform space and f a mapping of X into a set Y. Then the image $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ of $\mathcal{U}$ with respect to $f \times f$ is the finest fuzzy uniform structure on Y such that the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous.*

Proof. Obviously, the image $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ of the fuzzy filter $\mathcal{U}$ is a fuzzy filter on $Y \times Y$. From condition (U1) for $\mathcal{U}$, we have $(x, x)^\bullet \leqslant \mathcal{U}$ for all $x \in X$ and since $\mathcal{F}_L(f \times f)(x, x)^\bullet = (f(x), f(x))^\bullet$, then $\bigwedge_{y \in Y} (y, y)^\bullet \leqslant \bigwedge_{x \in X} \mathcal{F}_L(f \times f)(x, x)^\bullet$ and hence $(y, y)^\bullet \leqslant \mathcal{V}$ for all $y \in Y$. This means $\mathcal{V}$ fulfills the condition (U1) of a fuzzy uniform structure.

From condition (U2) for $\mathcal{U}$, it follows for each $v \in L^{Y \times Y}$ that

$$\begin{aligned}\mathcal{V}(v) &= \mathcal{U}(v \circ (f \times f)) = \mathcal{U}((v \circ (f \times f))^{-1}) \\ &= \mathcal{U}(v^{-1} \circ (f \times f)) = \mathcal{V}(v^{-1}) = \mathcal{V}^{-1}(v).\end{aligned}$$

Hence, $\mathcal{V}$ fulfills condition (U2).

Since $(l \circ (f \times f)) \diamond (k \circ (f \times f)) \leqslant w \circ (f \times f)$ implies $l \diamond k \leqslant w$ for all $k, l, w \in L^{Y \times Y}$ and since for all $u, v \in L^{X \times X}$ there are $k, l \in L^{Y \times Y}$ such that $u \leqslant k \circ (f \times f)$ and $v \leqslant l \circ (f \times f)$, then we get

$$\begin{aligned}\mathcal{F}_L(f \times f)(\mathcal{U} \diamond \mathcal{U})(w) &= \bigvee_{v \diamond u \leqslant w \circ (f \times f)} (\mathcal{U}(u) \wedge \mathcal{U}(v)) \\ &\leqslant \bigvee_{(l \circ (f \times f)) \diamond (k \circ (f \times f)) \leqslant w \circ (f \times f)} (\mathcal{U}(k \circ (f \times f)) \wedge \mathcal{U}(l \circ (f \times f))) \\ &\leqslant \bigvee_{l \diamond k \leqslant w} (\mathcal{F}_L(f \times f)\mathcal{U}(k) \wedge \mathcal{F}_L(f \times f)\mathcal{U}(l)) \\ &= (\mathcal{F}_L(f \times f)\mathcal{U} \diamond \mathcal{F}_L(f \times f)\mathcal{U})(w)\end{aligned}$$

and from condition (U3) for $\mathcal{U}$ we get $\mathcal{F}_L(f \times f)\mathcal{U} \diamond \mathcal{F}_L(f \times f)\mathcal{U} \leqslant \mathcal{F}_L(f \times f)(\mathcal{U})$. Hence, $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ fulfills condition (U3) and therefore it is a fuzzy uniform structure on Y such that, from the definition of $\mathcal{V}$, the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous.

It is clear that for any fuzzy uniform structure $\mathcal{W}$ on Y for which the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{W})$ is fuzzy uniformly continuous, we have $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U} \leqslant \mathcal{W}$. Therefore, $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ is the finest fuzzy uniform structure on Y such that $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous. Moreover, $\mathcal{U}$ is finer than the initial fuzzy uniform structure $\mathcal{F}_L^{-}(f \times f)\mathcal{V}$ of $\mathcal{V}$ with respect to f. □

For any class I we have the following result.

Proposition 2.2. *Let $((X_i, \mathcal{U}_i))_{i \in I}$ be a family of fuzzy uniform spaces and $(f_i)_{i \in I}$ a family of mappings f_i of sets X_i into a set X and let $\mathcal{V}_i = \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i$ for each $i \in I$. Then*

$$\mathcal{U} = \bigvee_{i \in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i = \bigvee_{i \in I} \mathcal{V}_i$$

is the finest fuzzy uniform structure on X for which each $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U})$ is fuzzy uniformly continuous.

Proof. For each $i \in I$ we have, by means of Proposition 2.1, that $\mathcal{V}_i$ is a fuzzy uniform structure on X. From condition (U1) for $\mathcal{V}_i$ it follows that $(x, x)^\bullet \leqslant \mathcal{V}_i$ for each $x \in X$ and hence $(x, x)^\bullet \leqslant \bigvee_{i \in I} \mathcal{V}_i = \mathcal{U}$. This means $\mathcal{U}$ fulfills the condition (U1) for a fuzzy uniform structure.

Since each $\mathcal{V}_i$ fulfills condition (U2), then for each $u \in L^{X \times X}$ we get

$$\begin{aligned}\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1}) &= \bigwedge_{i \in I} (\mathcal{V}_i(u^{-1})) = \bigwedge_{i \in I} (\mathcal{V}_i^{-1}(u)) \\ &= \bigwedge_{i \in I} (\mathcal{V}_i(u)) = \mathcal{U}(u).\end{aligned}$$

Hence, $\mathcal{U}$ fulfills condition (U2).

Condition (U3) for $\mathcal{V}_i$ for each $i \in I$ implies that

$$\mathcal{U} \circ \mathcal{U} = \bigvee_{i \in I} (\mathcal{V}_i \circ \mathcal{V}_i) \leqslant \bigvee_{i \in I} \mathcal{V}_i = \mathcal{U}.$$

Therefore, $\mathcal{U}$ fulfills condition (U3) and is then a fuzzy uniform structure on X.

It is clear that $\mathcal{F}_L(f_i \times f_i)\mathcal{U}_i = \mathcal{V}_i \leqslant \mathcal{U}$ holds for each $i \in I$. Thus each $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U})$ is fuzzy uniformly continuous. Moreover, $\mathcal{U} = \bigvee_{i \in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i$ is finer than any fuzzy uniform structure $\mathcal{W}$ on X for which each $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{W})$ is fuzzy uniformly continuous. This means $\mathcal{U}$ is the finest fuzzy uniform structure on X such that each $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U})$ is fuzzy uniformly continuous. We also have that each $\mathcal{U}_i$ is finer than the initial fuzzy uniform structure $\mathcal{F}_L^-(f_i \times f_i)\mathcal{U}$ of $\mathcal{U}$ with respect to f_i. □

2.1. *Final lifts in* FUN

We showed in [2] that the category FUN of fuzzy uniform spaces is topological and this means the final lifts also exist.

Let I be any class. For each $i \in I$ let $(X_i, \mathcal{U}_i)$ be a fuzzy uniform space and f_i a mapping of sets X_i into a set X. For any fuzzy uniform structure $\mathcal{U}$ on X, for which all mappings $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U})$ are fuzzy uniformly continuous, the family $(f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U}))_{i \in I}$ is called a *final lift* of $(f_i : X_i \to X,\ \mathcal{U}_i)_{i \in I}$ provided for any fuzzy uniform space $(Y, \mathcal{V})$ and mapping $f : X \to Y$, $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous if and only if for all $i \in I$ the mappings $f \circ f_i : (X_i, \mathcal{U}_i) \to (Y, \mathcal{V})$ are fuzzy uniformly continuous.

In each final lift $(f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U}))_{i \in I}$ of $(f_i : X_i \to X,\ \mathcal{U}_i)_{i \in I}$, we easily get that $\mathcal{U}$ is the finest fuzzy uniform structure on X for which each $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U})$ is fuzzy uniformly continuous. The converse, that is, the finest fuzzy uniform structures, characterized in Propositions 2.1 and 2.2, provide final lifts will be proved in the following proposition.

Proposition 2.3. *Let $\mathcal{U}$ be the fuzzy uniform structure on X, given in Proposition* 2.2. *Then $(f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U}))_{i \in I}$ is the final lift of $(f_i : X_i \to X,\ \mathcal{U}_i)_{i \in I}$.*

Proof. Let $(Y, \mathcal{V})$ be a fuzzy uniform space and $f : X \to Y$ a mapping. From Proposition 2.2 each $f_i : (X_i, \mathcal{U}_i) \to (X, \mathcal{U})$ is fuzzy uniformly continuous and hence, if $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous, then each mapping $f \circ f_i : (X_i, \mathcal{U}_i) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous.

Now, let $f \circ f_i : (X_i, \mathcal{U}i) \to (Y, \mathcal{V})$ be fuzzy uniformly continuous for all $i \in I$. Then $\bigvee_{i \in I} \mathcal{F}_L(f \circ f_i \times f \circ f_i)\mathcal{U}_i$ is, by means of Propositions 2.1 and 2.2, the finest fuzzy uniform structure on Y for which each $f \circ f_i$ is fuzzy uniformly continuous. Since $\mathcal{F}_L$ is a covariant functor from $\mathcal{F}_L(X \times X)$ into $\mathcal{F}_L(Y \times Y)$, then

$$\begin{aligned}\mathcal{V} &\geqslant \bigvee_{i \in I} \mathcal{F}_L(f \circ f_i \times f \circ f_i)\mathcal{U}_i \;=\; \bigvee_{i \in I} (\mathcal{F}_L(f \times f) \circ \mathcal{F}_L(f_i \times f_i))\mathcal{U}_i \\ &= \bigvee_{i \in I} \mathcal{F}_L(f \times f)(\mathcal{F}_L(f_i \times f_i)\mathcal{U}_i) \;=\; \mathcal{F}_L(f \times f)\left(\bigvee_{i \in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i\right).\end{aligned}$$

Hence, $f : (X, \mathcal{U} = \bigvee_{i \in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous. □

A *final fuzzy uniform structure* is the fuzzy uniform structure which provide final lift [1,19]. From Proposition 2.3 we get that the finest fuzzy uniform structures, defined in Propositions 2.1 and 2.2, coincide with the final fuzzy uniform structures. That is, if $f : X \to Y$ is a mapping and $\mathcal{U}$ a fuzzy uniform structure on X, then $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ is the final fuzzy uniform structure of $\mathcal{U}$ with respect to f. Moreover, in case of a family $(\mathcal{U}_i)_{i \in I}$ of fuzzy uniform structures $\mathcal{U}_i$ on X_i and a family $(f_i)_{i \in I}$ of mappings f_i of X_i into a set X, $\mathcal{U} = \bigvee_{i \in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i$ is the final fuzzy uniform structure of $(\mathcal{U}_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$.

2.2. *Fuzzy uniform quotient spaces and fuzzy uniform sum spaces*

The fuzzy uniform quotient spaces and the fuzzy uniform sum spaces, in the categorical sense, are special final fuzzy uniform spaces [1] and therefore these spaces are examples on final fuzzy uniform spaces and can be characterized as follows: let $(X, \mathcal{U})$ be a fuzzy uniform space and $f : X \to Y$ a surjective mapping. Then the fuzzy uniform quotient structure is the final fuzzy uniform structure $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ of $\mathcal{U}$ with respect to f and the pair $(Y, \mathcal{V})$ is the fuzzy uniform quotient space. Moreover, if for each element i of a set I, $(X_i, \mathcal{U}_i)$ is a fuzzy uniform space, $X = \biguplus_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\})$ the disjoint union of the family $(X_i)_{i \in I}$ and $e_i : X_i \to X$, for each $i \in I$ the related canonical injection defined by: $e_i(x_i) = (x_i, i)$, then the fuzzy uniform sum structure is the final fuzzy uniform structure $\mathcal{U} = \bigvee_{i \in I} \mathcal{F}_L(e_i \times e_i)\mathcal{U}_i$ of $(\mathcal{U}_i)_{i \in I}$ with respect to $(e_i)_{i \in I}$ and the pair $(X, \mathcal{U})$ is the fuzzy uniform sum space.

3. Final global fuzzy neighborhood structures

In this section we characterize the final global fuzzy neighborhood structure and then we study the relation between the final global fuzzy neighborhood structure and the final fuzzy uniform structure.

For I being a singleton we have the following result.

Proposition 3.1. *Assume that h is a global fuzzy neighborhood structure on a set X and f is a mapping of X into a set Y. Then the mapping $k : \mathcal{F}_L Y \to \mathcal{F}_L Y$ defined by*

$$k = \mathcal{F}_L f \circ h \circ \mathcal{F}_L^- f \tag{3.1}$$

is the finest global fuzzy neighborhood structure on Y for which the mapping $f : (X, h) \to (Y, k)$ is (h, k)-continuous.

Proof. For each fuzzy filter $\mathcal{N}$ on Y there is a fuzzy filter $\mathcal{M}$ on X such that $\mathcal{F}_L(\mathcal{M}) \leqslant \mathcal{N}$. Hence, $k(\mathcal{N})$ exists. From that h is a hull operator, it follows that $\mathcal{N} \leqslant k(\mathcal{N})$ for each $\mathcal{N} \in \mathcal{F}_L Y$. Therefore, k fulfills condition (N1) of a global fuzzy neighborhood structure.

Each $\mathcal{F}_L f$, h and $\mathcal{F}_L^- f$ preserve finite suprema of fuzzy filters, then k also preserves finite suprema and hence k fulfills condition (N2).

The proof that k fulfills the condition (N3) follows from the properties of $\mathcal{F}_L f$ and $\mathcal{F}_L^- f$ and from that h fulfills condition (N3).

Because of that h fulfills condition (N4), we get the following:

$$\begin{aligned} k(\mathcal{N})(g) &= \mathcal{F}_L f(h\mathcal{F}_L^- f(\mathcal{N}))(g) = h(\mathcal{F}_L^- f(\mathcal{N}))(g \circ f) \\ &= \mathcal{F}_L^- f(\mathcal{N})(x \mapsto h(\dot{x})(g \circ f)) = \bigvee_{w \circ f \leqslant v} \mathcal{N}(w), \end{aligned}$$

where $v \in L^X$ defined by $v(x) = h(\dot{x})(g \circ f)$. Hence, k also fulfills condition (N4). Therefore, k is a global fuzzy neighborhood structure on Y.

From that $\mathcal{M} \leqslant \mathcal{F}_L^- f(\mathcal{F}_L f(\mathcal{M}))$ for all $\mathcal{M} \in \mathcal{F}_L X$ it follows $\mathcal{F}_L f \circ h \leqslant k \circ \mathcal{F}_L f$ and then $f : (X, h) \to (Y, k)$ is (h, k)-continuous.

Now assume that l is any global fuzzy neighborhood structure on Y for which f is (h, l)-continuous. For each fuzzy filter $\mathcal{N}$ on Y we have $\mathcal{F}_L f(\mathcal{F}_L^- f(\mathcal{N})) \leqslant \mathcal{N}$ holds and thus,

$$k(\mathcal{N}) = \mathcal{F}_L f(h(\mathcal{F}_L^- f(\mathcal{N}))) \leqslant l(\mathcal{F}_L f(\mathcal{F}_L^- f(\mathcal{N}))) \leqslant l(\mathcal{N}).$$

Hence, k is the finest global fuzzy neighborhood structure on Y for which f is (h, k)-continuous. We also have that $h(\mathcal{M}) \leqslant (\mathcal{F}_L^- f \circ k \circ \mathcal{F}_L f)(\mathcal{M})$ holds for all $\mathcal{M} \in \mathcal{F}_L X$, that is, h is finer than the initial global fuzzy neighborhood structure $\mathcal{F}_L^- f \circ k \circ \mathcal{F}_L f$ of k with respect to f. □

Now, let I be a class and for each $i \in I$ let (X_i, h_i) be a global fuzzy neighborhood space and f_i a mapping of X_i into a set X.

Proposition 3.2. *For each $i \in I$ let $k_i = \mathcal{F}_L f_i \circ h_i \circ \mathcal{F}_L^- f_i$. Then the mapping $h : \mathcal{F}_L X \to \mathcal{F}_L X$, defined by*

$$h(\mathcal{M}) = \bigvee_{i \in I} k_i = \bigvee_{i \in I} (k_i(\mathcal{M}))$$

for each $\mathcal{M} \in \mathcal{F}_L X$, is the finest global fuzzy neighborhood structure on X such that each $f_i : (X_i, h_i) \to (X, h)$ is (h_i, h)-continuous.

Proof. Proposition 3.1 implies that for each $i \in I$, k_i is a global fuzzy neighborhood structure on X_i. That is, k_i fulfills conditions (N1) and (N2) of a global fuzzy neighborhood structure and hence h also fulfills conditions (N1) and (N2).

Since each k_i is idempotent, we have

$$\bigvee_{i \in I} k_i \left(\bigvee_{j \in I} k_j(\mathcal{M}) \right) = \bigvee_{i,j \in I} k_i(k_j(\mathcal{M})) \leqslant \bigvee_{i \in I} k_i(k_i(\mathcal{M})) = \bigvee_{i \in I} k_i(\mathcal{M}).$$

Hence, $h(h(\mathcal{M})) \leqslant h(\mathcal{M})$ and since h is a hull operator, we get $h(h(\mathcal{M})) = h(\mathcal{M})$ for each $\mathcal{M} \in \mathcal{F}_L X$. This means that h fulfills condition (N3).

Moreover, for each $i \in I$ we have $k_i(\mathcal{M})(g) \leqslant \mathcal{M}(\mathrm{int}_{k_i} g)$ for each $g \in L^X$, and each $\mathcal{M} \in \mathcal{F}_L X$ and hence

$$h(\mathcal{M})(g) = \left(\bigvee_{i \in I} k_i(\mathcal{M}) \right)(g) = \bigwedge_{i \in I} (k_i(\mathcal{M})(g)) \leqslant k_i(\mathcal{M})(g) \leqslant \mathcal{M}(\mathrm{int}_h g),$$

where int_{k_i} and int_h are the interior operators with respect to k_i and h, respectively. Thus, h fulfills condition (N4).

For each $i \in I$ and each $\mathcal{M} \in \mathcal{F}_L X$ we have $\mathcal{F}_L f_i(h_i(\mathcal{M})) \leqslant h(\mathcal{F}_L f_i(\mathcal{M}))$ and then f_i is (h_i, h)-continuous. Let l be any global fuzzy neighborhood structure on X such that each f_i is (h_i, l)-continuous, that is, $\mathcal{F}_L f_i \circ h_i \leqslant l \circ \mathcal{F}_L f_i$ holds for all $i \in I$. Hence,

$$h(\mathcal{M}) = \left(\bigvee_{i \in I} (\mathcal{F}_L f_i \circ h_i \circ \mathcal{F}_L^- f_i) \right)(\mathcal{M}) \leqslant l(\mathcal{M}).$$

That is, h is the finest global fuzzy neighborhood structure on X for which f_i is (h_i, h)-continuous. Moreover, we get that $h_i \leqslant \mathcal{F}_L^- f_i \circ h \circ \mathcal{F}_L f_i$ holds and therefore for each $i \in I$, h_i is finer than the initial global fuzzy neighborhood structure $\mathcal{F}_L^- f_i \circ h \circ \mathcal{F}_L f_i$ of h with respect to f_i. □

3.1. Final lifts in FNS

It is shown in [6] that the final lifts in the category FNS of global fuzzy neighborhood spaces exist. In the following proposition we show that the global fuzzy neighborhood structures, characterized in Propositions 3.1 and 3.2, provide final lifts.

Proposition 3.3. *Let h be the global fuzzy neighborhood structure on X, given in Proposition* 3.2. *Then $(f_i : (X_i, h_i) \to (X, h))_{i \in I}$ is the final lift of $(f_i : X_i \to X,\ h_i)_{i \in I}$.*

Proof. Let (Y, k) be a global fuzzy neighborhood space and $f : X \to Y$ a mapping. If $f : (X, h) \to (Y, k)$ is (h, k)-continuous, then by means of Proposition 3.2 each mapping $f_i : (X_i, h_i) \to (X, h)$ is (h_i, h)-continuous and thus each mapping $f \circ f_i : (X_i, h_i) \to (Y, k)$ is (h_i, k)-continuous.

Let $f \circ f_i : (X_i, h_i) \to (Y, k)$ be (h_i, k)-continuous for all $i \in I$. Then from Propositions 3.1 and 3.2 it follows that

$$\bigvee_{i \in I} (\mathcal{F}_L(f \circ f_i) \circ h_i \circ \mathcal{F}_L^-(f \circ f_i)) \leqslant k$$

and hence from Proposition 1.1 and from that $\mathcal{F}_L f$ preserve suprema of fuzzy filters we get that

$$\mathcal{F}_L f \circ h \circ \mathcal{F}_L^- f_i = \mathcal{F}_L f \circ \bigvee_{i \in I} (\mathcal{F}_L f_i \circ h_i \circ \mathcal{F}_L^- f_i) \circ \mathcal{F}_L^- f \leqslant k.$$

Since $f : (X, h) \to (Y, \mathcal{F}_L f \circ h \circ \mathcal{F}_L^- f_i)$ is $(h, \mathcal{F}_L f \circ h \circ \mathcal{F}_L^- f_i)$-continuous, then $f : (X, h) \to (Y, k)$ is (h, k)-continuous. □

Since the final global fuzzy neighborhood structure is the global fuzzy neighborhood structure which provide final lift, then Proposition 3.3 implies that the finest global fuzzy neighborhood structures, defined in Propositions 3.1 and 3.2, coincide with the final global fuzzy neighborhood structures.

Remark 3.1. If each h_i is a global homogeneous fuzzy neighborhood structure, then the final global fuzzy neighborhood structure of $(h_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ also is homogeneous.

In the next lines, we show that the global homogeneous fuzzy neighborhood structure associated with the final fuzzy uniform structure of a family $(\mathcal{U}_i)_{i\in I}$ of fuzzy uniform structures with respect to a family $(f_i)_{i\in I}$ of mappings f_i of X_i into X coincides with the final global fuzzy neighborhood structure of the family $(h_{\mathcal{U}_i})_{i\in I}$ of global homogeneous fuzzy neighborhood structures $h_{\mathcal{U}_i}$ associated with $\mathcal{U}_i$. To verify this relation we need the following results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping and $\mathcal{U}$ a fuzzy filter on $X \times X$. Then for each fuzzy filter $\mathcal{M}$ on Y and $g \in L^Y$ we have $\mathcal{U}[\mathcal{F}_L^- f(\mathcal{M})](g \circ f)$ is less than $(\mathcal{F}_L(f \times f)\mathcal{U})[\mathcal{M}](g)$.*

Proof. For each $g \in L^Y$ we have

$$\begin{aligned}(\mathcal{F}_L(f \times f)\mathcal{U})[\mathcal{M}](g) &= \bigvee_{v[k]\leqslant g} (\mathcal{F}_L(f \times f)\mathcal{U}(v) \wedge \mathcal{M}(k))\\ &= \bigvee_{v[k]\leqslant g} (\mathcal{U}(v \circ (f \times f)) \wedge \mathcal{M}(k))\\ &= \bigvee_{u[k]\leqslant g\circ f} (\mathcal{U}(u) \wedge \mathcal{M}(k)) \geqslant \mathcal{U}[\mathcal{F}_L^- f(\mathcal{M})](g \circ f). \qquad \square\end{aligned}$$

Lemma 3.2. *For each $\mathcal{V}_i \in \mathcal{F}_L(X \times X)$ and each $\mathcal{M} \in \mathcal{F}_L X$ we get that $(\bigvee_{i\in I} \mathcal{V}_i)[\mathcal{M}]$ is finer than $\bigvee_{i\in I}(\mathcal{V}_i[\mathcal{M}])$.*

Proof. For each $g \in L^X$ we have

$$\begin{aligned}\left(\bigvee_{i\in I} \mathcal{V}_i[\mathcal{M}]\right)(g) &= \bigwedge_{i\in I}(\mathcal{V}_i[\mathcal{M}](g)) = \bigwedge_{i\in I}\left(\bigvee_{u[k]\leqslant g} (\mathcal{V}_i(u) \wedge \mathcal{M}(k))\right)\\ &\leqslant \bigvee_{u[k]\leqslant g}\bigwedge_{i\in I}(\mathcal{V}_i(u)) \wedge \mathcal{M}(k) = \bigvee_{u[k]\leqslant g}\left(\left(\bigvee_{i\in I}\mathcal{V}_i\right)(u) \wedge \mathcal{M}(k)\right)\\ &= \left(\bigvee_{i\in I}\mathcal{V}_i\right)[\mathcal{M}](g). \qquad \square\end{aligned}$$

First, we study the case of one mapping.

Proposition 3.4. *Let $h_{\mathcal{U}}$ the global homogeneous fuzzy neighborhood structure associated with the fuzzy uniform structure $\mathcal{U}$ on a set X and $\mathcal{V}$ the final fuzzy uniform structure of $\mathcal{U}$ with respect to the mapping f of X into a set Y. Then the global homogeneous fuzzy neighborhood structure $h_{\mathcal{V}}$ associated with $\mathcal{V}$ coincides with the final global fuzzy neighborhood structure $\mathcal{F}_L f \circ h_{\mathcal{U}} \circ \mathcal{F}_L^- f$ of $h_{\mathcal{U}}$ with respect to f.*

Proof. From Proposition 1.3 it follows that the mapping $f : (X, h_{\mathcal{U}}) \to (Y, h_{\mathcal{V}})$ is $(h_{\mathcal{U}}, h_{\mathcal{V}})$-continuous and thus $\mathcal{F}_L f \circ h_{\mathcal{U}} \circ \mathcal{F}_L^- f$ is finer than $h_{\mathcal{V}}$.

From Lemma 3.1 and from that $\mathcal{V} = \mathcal{F}_L(f \times f)\mathcal{U}$ for each fuzzy filter $\mathcal{N}$ on Y and for each $g \in L^Y$, we have

$$(\mathcal{F}_L f \circ h_{\mathcal{U}} \circ \mathcal{F}_L^- f)(\mathcal{N})(g) = \mathcal{U}[\mathcal{F}_L^- f(\mathcal{N})](g \circ f) \leqslant \mathcal{V}[\mathcal{N}](g) = h_{\mathcal{V}}(\mathcal{N})(g).$$

Hence, $\mathcal{F}_L f \circ h_{\mathcal{U}} \circ \mathcal{F}_L^- f$ is coarser than $h_{\mathcal{V}}$. $\square$

Now, let I be any class.

Proposition 3.5. *The global homogeneous fuzzy neighborhood structure $h_{\mathcal{U}}$ associated with the final fuzzy uniform structure $\mathcal{U}$ of $(\mathcal{U}_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ coincides with the final global fuzzy neighborhood structure $\bigvee_{i\in I}(\mathcal{F}_L f_i \circ h_{\mathcal{U}_i} \circ \mathcal{F}_L^- f_i)$ of the family $(h_{\mathcal{U}_i})_{i\in I}$ of global homogeneous fuzzy neighborhood structures $h_{\mathcal{U}_i}$ associated with $\mathcal{U}_i$.*

Proof. We have that each $f_i : (X_i, h_{\mathcal{U}_i}) \to (X, h_{\mathcal{U}})$ is $(h_{\mathcal{U}_i}, h_{\mathcal{U}})$-continuous. Hence, $\bigvee_{i\in I}(\mathcal{F}_L f_i \circ h_{\mathcal{U}_i} \circ \mathcal{F}_L^- f_i)$ is finer than $h_{\mathcal{U}}$.

Moreover, if $\mathcal{U} = \bigvee_{i\in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i = \bigvee_{i\in I} \mathcal{V}_i$, then from Lemma 3.2 we have $(\bigvee_{i\in I} \mathcal{V}_i)[\mathcal{M}]$ is finer than $\bigvee_{i\in I}(\mathcal{V}_i[\mathcal{M}])$ for each $\mathcal{M} \in \mathcal{F}_L X$. Thus, for each $g \in L^X$ we have

$$\left(\bigvee_{i\in I}(\mathcal{F}_L f_i \circ h_{\mathcal{U}_i} \circ \mathcal{F}_L^- f_i)\right)(\mathcal{M})(g) = \bigwedge_{i\in I}((\mathcal{F}_L f_i \circ h_{\mathcal{U}_i} \circ \mathcal{F}_L^- f_i)(\mathcal{M})(g))$$

and hence

$$\left(\bigvee_{i\in I}(\mathcal{F}_L f_i \circ h_{\mathcal{U}_i} \circ \mathcal{F}_L^- f_i)\right)(\mathcal{M})(g) \leqslant h_{\mathcal{V}_i}(\mathcal{M})(g) \leqslant h_{\mathcal{U}}(\mathcal{M})(g).$$

This means $\bigvee_{i\in I}(\mathcal{F}_L f_i \circ h_{\mathcal{U}_i} \circ \mathcal{F}_L^- f_i)$ is coarser than $h_{\mathcal{U}}$. □

4. Final fuzzy topologies

Here, we show the relation between the final fuzzy uniform structure (final global fuzzy neighborhood structure) and the final fuzzy topology. The expected relation between the initial global fuzzy neighborhood structure and the initial fuzzy topology also is verified.

For a family $((X_i, \tau_i))_{i\in I}$ of fuzzy topological spaces and a family $((f_i))_{i\in I}$ of mappings f_i of X_i into a set X, the infimum $\tau = \bigwedge_{i\in I} f_i(\tau_i) = \bigcap_{i\in I} f_i(\tau_i)$, where $f_i(\tau_i) = \{g \in L^X \mid g \circ f_i \in \tau_i\}$, is the finest fuzzy topology on X for which all mappings $f_i : (X_i, \tau_i) \to (X, \tau)$ are fuzzy continuous [17]. It easily seen that $\bigwedge_{i\in I} f_i(\tau_i)$ fulfills the requirements of a final lift in the category of fuzzy topological spaces and hence $\bigwedge_{i\in I} f_i(\tau_i)$ is the final fuzzy topology of $(\tau_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ and in particular, $f_i(\tau_i)$ is the final fuzzy topology of τ_i with respect to f_i.

In the following will be shown that the fuzzy topology associated with the final fuzzy uniform structure of a family $(\mathcal{U}_i)_{i\in I}$ of fuzzy uniform structures $\mathcal{U}_i$ coincides with the final fuzzy topology of the family $(\tau_{\mathcal{U}_i})_{i\in I}$ of fuzzy topologies $\tau_{\mathcal{U}_i}$ associated with $\mathcal{U}_i$.

First, consider the case of one mapping.

Proposition 4.1. *Let $(X, \mathcal{U})$ be a fuzzy uniform space, f a mapping of X onto a set Y, $\mathcal{V}$ the final fuzzy uniform structure of $\mathcal{U}$ with respect to f and $\tau_{\mathcal{U}}$, $\tau_{\mathcal{V}}$ the fuzzy topologies associated with $\mathcal{U}$, $\mathcal{V}$, respectively. Then $\tau_{\mathcal{V}}$ coincides with the final fuzzy topology $f(\tau_{\mathcal{U}})$ of $\tau_{\mathcal{U}}$ with respect to f.*

Proof. From Propositions 1.2 and 2.1 it follows that $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ is fuzzy continuous and hence $f(\tau_{\mathcal{U}})$ is finer than $\tau_{\mathcal{V}}$.

If $g \in f(\tau_{\mathcal{U}})$, then $g \circ f \in \tau_{\mathcal{U}}$. Since $(v \circ (f \times f))[l \circ f] \leqslant g \circ f$ implies $v[l] \leqslant g$ for all $v \in L^{Y\times Y}$ and all $l \in L^Y$, and since for all $u \in L^{X\times X}$ and all $k \in L^X$ there are $v \in L^{Y\times Y}$ and $l \in L^Y$ such that $u \leqslant v \circ (f \times f)$ and $k \leqslant l \circ f$, then from that f is surjective it follows for all $y \in Y$ there is $x \in X$ such that $g(y) = g(f(x)) = \mathcal{U}[\dot{x}](g \circ f)$ and therefore

$$\begin{aligned} g(y) &= \bigvee_{u[k]\leqslant g\circ f} (\mathcal{U}(u) \wedge k(x)) \leqslant \bigvee_{(v\circ(f\times f))[l\circ f]\leqslant g\circ f} (\mathcal{U}(v \circ (f \times f)) \wedge (l \circ f)(x)) \\ &\leqslant \bigvee_{v[l]\leqslant g} (\mathcal{F}_L(f \times f)\mathcal{U}(v) \wedge l(y)) = \mathcal{V}[\dot{y}](g) = (\mathrm{int}_{\mathcal{V}} g)(y), \end{aligned}$$

where $\mathrm{int}_{\mathcal{V}}$ is the interior operator with respect to $\tau_{\mathcal{V}}$. Hence, $g \in \tau_{\mathcal{V}}$ and thus $f(\tau_{\mathcal{U}})$ is coarser than $\tau_{\mathcal{V}}$. □

For any class I, let $(f_i)_{i\in I}$ be a family of mappings f_i of sets X_i onto a set X, and for each $i \in I$ let $\mathcal{U}_i$ be a fuzzy uniform structure on X_i.

Proposition 4.2. *If $\mathcal{U}$ is the final fuzzy uniform structure of $(\mathcal{U}_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ and $\tau_{\mathcal{U}_i}$ the fuzzy topology associated with $\mathcal{U}_i$, then the fuzzy topology $\tau_{\mathcal{U}}$ associated with $\mathcal{U}$ coincides with the final fuzzy topology $\bigwedge_{i\in I} f_i(\tau_{\mathcal{U}_i})$ of the family $(\tau_{\mathcal{U}_i})_{i\in I}$ with respect to $(f_i)_{i\in I}$.*

Proof. Similarly as in the proof of Proposition 4.1, we get that $\bigwedge_{i\in I} f_i(\tau_{\mathcal{U}_i})$ is finer than $\tau_{\mathcal{U}}$.

Let $g \in \bigwedge_{i\in I} f_i(\tau_{\mathcal{U}_i})$. Then $g \in f_i(\tau_{\mathcal{U}_i})$ for each $i \in I$ and hence, by means of Proposition 4.1, we get $g \in \tau_{\mathcal{F}_L(f_i\times f_i)\mathcal{U}_i}$ for each $i \in I$. Since $\mathcal{U} = \bigvee_{i\in I} \mathcal{F}_L(f_i \times f_i)\mathcal{U}_i$, then $g \in \tau_{\mathcal{U}}$ and thus $\bigwedge_{i\in I} f_i(\tau_{\mathcal{U}_i})$ is coarser than $\tau_{\mathcal{U}}$. □

To show that the fuzzy topology associated with the final global fuzzy neighborhood structure h of a family $(h_i)_{i\in I}$ of global fuzzy neighborhood structures h_i with respect to a family $(f_i)_{i\in I}$ of mappings f_i of sets X_i onto a set X coincides with the final fuzzy topology of the family $(\tau_{h_i})_{i\in I}$ of fuzzy topologies τ_{h_i} associated with h_i we need the following result.

Proposition 4.3 (*Gähler et al. [6]*). *Let $f : (X, h) \to (Y, k)$ be a (h, k)-continuous mapping between global fuzzy neighborhood spaces. Then the mapping $f : (X, \tau_h) \to (Y, \tau_k)$ between the associated fuzzy topological spaces is fuzzy continuous.*

For I being a singleton we get the following result.

Proposition 4.4. *Let h be a global fuzzy neighborhood structure on a set X, f a mapping of X onto a set Y and k the final global fuzzy neighborhood structure of h with respect to f and let τ_h be the fuzzy topology associated with h. Then the fuzzy topology τ_k associated with k coincides with the final fuzzy topology $f(\tau_h)$* of *τ_h with respect to f.*

Proof. From Proposition 4.3 we get that $f : (X, \tau_h) \to (Y, \tau_k)$ is fuzzy continuous and hence $f(\tau_h)$ is finer than τ_k.

If $g \in f(\tau_h)$, then $g \circ f \in \tau_h$ and since f is surjective, then for each $y \in Y$ there is $x \in X$ such that $g(y) = g(f(x))$ and thus we get the following:

$$g(y) = g(f(x)) = h(\dot{x})(g \circ f) = \mathcal{F}_L f(h(\dot{x}))(g) \leqslant k(\dot{y})(g).$$

That is, $g \in \tau_k$ and hence τ_k is finer than $f(\tau_h)$. □

For any class I we get the following result.

Proposition 4.5. *The fuzzy topology τ_h associated with the final global fuzzy neighborhood structure h of $(h_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ coincides with the final fuzzy topology $\bigwedge_{i\in I} f_i(\tau_{h_i})$ of the family $(\tau_{h_i})_{i\in I}$ of fuzzy topologies associated with h_i.*

Proof. For each $i \in I$, we have $f_i : (X, \tau_h) \to (X_i, \tau_{h_i})$ is fuzzy continuous and hence, $\bigwedge_{i\in I} f_i(\tau_{h_i})$ is finer than τ_h.

Let $g \in \bigwedge_{i\in I} f_i(\tau_{h_i})$ be hold and let $k_i = \mathcal{F}_L f_i \circ h_i \circ \mathcal{F}_L^- f_i$. Then $h = \bigvee_{i\in I} k_i$ and from Proposition 4.4, it follows that $g \in \tau_{k_i}$ and hence

$$h(\dot{x})(g) = \bigwedge_{i\in I} (k_i(\dot{x})(g)) = \bigwedge_{i\in I} g(x) = g(x).$$

Hence, $g \in \tau_h$. Thus τ_h is finer than $\bigwedge_{i\in I} (f_i \tau_{h_i})$. □

Now, we are going to show that the initial global fuzzy neighborhood structure has a similar relation with the initial fuzzy topology, that is, the fuzzy topology associated with the initial global fuzzy neighborhood structure h of a family $(h_i)_{i\in I}$ of global fuzzy neighborhood structures h_i on sets X_i with respect to a family $(f_i)_{i\in I}$ of mappings f_i of a set X into sets X_i coincides with the initial fuzzy topology of the family $(\tau_{h_i})_{i\in I}$ of fuzzy topologies τ_{h_i} associated with h_i. Recall that the initial global fuzzy neighborhood structure h of $(h_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ is $(\bigwedge_{i\in I}(\mathcal{F}_L^- f_i \circ h_i \circ \mathcal{F}_L f_i))^{\vee}$, where $\mathcal{F}_L^- f_i \circ h_i \circ \mathcal{F}_L f_i$ is the initial global fuzzy neighborhood structure of h_i with respect to f_i (see [6]). Moreover, for a family of fuzzy topological spaces $((X_i, \tau_i))_{i\in I}$, the initial fuzzy topology τ of

$(\tau_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ is $\bigvee_{i\in I} f_i^{-1}(\tau_i)$, where $f_i^{-1}\tau_i = \{g \circ f_i \mid g \in \tau_i\}$ is the initial fuzzy topology of τ_i with respect to f_i (see [2,9,21]).

First, consider the case of *I* being a singleton.

Proposition 4.6. *Let (Y, k) be a global fuzzy neighborhood space and τ_k the fuzzy topology associated with k, and let h be the initial global fuzzy neighborhood structure of k with respect to a mapping f of a set X into Y. Then the fuzzy topology τ_h associated with h coincides with the initial fuzzy topology $f^{-1}(\tau_k)$ of τ_k with respect to f.*

Proof. Since $f : (X, h) \to (Y, k)$ is (h, k)-continuous, it follows from Proposition 4.3 that $f : (X, \tau_h) \to (Y, \tau_k)$ is fuzzy continuous and hence $f^{-1}(\tau_k)$ is coarser than τ_h.

Now, let $g \in \tau_h$ be hold.

$$\begin{aligned} g(x) = h(\dot{x})(g) &= (\mathcal{F}_L^- f k(\dot{fx}))(g) \\ &= \bigvee_{l\circ f \leqslant g} (k(\dot{fx}))(l) = \bigvee_{l\circ f \leqslant g} (\mathrm{int}_k l)(f(x)). \end{aligned}$$

Hence, $g \in f^{-1}(\tau_k)$ and hence τ_h is coarser than $f^{-1}(\tau_k)$. □

Now, let *I* be any class.

Proposition 4.7. *The fuzzy topology τ_h associated with the initial global fuzzy neighborhood structure h of $(h_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ coincides with the initial fuzzy topology $\bigvee_{i\in I} f_i^{-1}\tau_{h_i}$ of the family $(\tau_{h_i})_{i\in I}$ with respect to $(f_i)_{i\in I}$.*

Proof. Similarly as in the proof of Proposition 4.6 we get that $\bigvee_{i\in I} f_i^{-1}\tau_{h_i}$ is coarser than τ_h.

Let $g \in \tau_h$ be hold. Then for any positive integer *n*, any collection $g_1, \ldots, g_n$ of L^X such that $g_1 \wedge \cdots \wedge g_n \leqslant g$, we get the following:

$$\begin{aligned} g(x) = h(\dot{x})(g) &= \left(\bigwedge_{i\in I} k_i(\dot{x})\right)(g) \\ &= \bigvee_{g_1\wedge\cdots\wedge g_n \leqslant g} (k_1(\dot{x})(g_1) \wedge \cdots \wedge k_n(\dot{x})(g_n)) \\ &\leqslant \bigvee_{g_1\wedge\cdots\wedge g_n \leqslant g} \left(\bigvee_{l_1\circ f \leqslant g_1} f^{-1}(\mathrm{int}_{h_1} l_1) \wedge \cdots \wedge \bigvee_{l_n\circ f \leqslant g_n} f^{-1}(\mathrm{int}_{h_n} l_n)\right) \\ &= \bigwedge_{i=1}^{n} \left(\bigvee_{l_i\circ f_i \leqslant g_i} f^{-1}(\mathrm{int}_{h_i} l_i)\right). \end{aligned}$$

Hence, $g \in \bigvee_{i\in I} f_i^{-1}\tau_{h_i}$. Thus, τ_h is coarser than $\bigvee_{i\in I} f_i^{-1}\tau_{h_i}$. □

References

[1] J. Adámek, H. Herrlich, G. Strecker, Abstract and Concrete Categories, Wiley, New York, 1990.
[2] F. Bayoumi, On initial and final fuzzy uniform structures, Fuzzy Sets and Systems 133 (2003) 299–319.
[3] C.H. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[4] P. Eklund, W. Gähler, Fuzzy filter functors and convergence, in: Applications of Category Theory to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht, 1992, pp. 109–136.
[5] W. Gähler, The general fuzzy filter approach to fuzzy topology, I, Fuzzy Sets and Systems 76 (1995) 205–224.
[6] W. Gähler, F. Bayoumi, A. Kandil, A. Nouh, The theory of global fuzzy neighborhood structures, Part I, The general case, Fuzzy Sets and Systems 97 (1998) 225–254.
[7] W. Gähler, F. Bayoumi, A. Kandil, A. Nouh, The theory of global fuzzy neighborhood structures, III, Fuzzy uniform structures, Fuzzy Sets and Systems 98 (1998) 175–199.
[8] J.A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.

[9] J. Gutiérrez García, M. de Prada Vicente, A.P. Šostak, A unified approach to the concept of fuzzy L-uniform space, in: S.E. Rodabaugh, E.P. Klement (Eds.), Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic, vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, pp. 81–114.
[10] U. Höhle, Probabilistic uniformization of fuzzy uniformities, Fuzzy Sets and Systems 1 (1978) 311–332.
[11] U. Höhle, Characterization of L-topologies by L-valued neighborhoods, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, Boston, London, 1999, pp. 389–432.
[12] U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, Boston, London, 1999.
[13] U. Höhle, A. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, Boston, London, 1999, pp. 123–272.
[14] B. Hutton, Uniformities on fuzzy topological spaces, J. Math. Anal. Appl. 58 (1977) 559–571.
[15] J.R. Isbell, Uniform Spaces, Mathematical Surveys, vol. 12, American Mathematical Society, Providence, RI, 1964.
[16] W. Kotzé, Uniform spaces, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, Boston, London, 1999, pp. 553–580.
[17] R. Lowen, Initial and final fuzzy topologies and fuzzy Tychonoff theorem, J. Math. Anal. Appl. 58 (1977) 11–21.
[18] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981) 370–385.
[19] G. Richter, Kategorielle Algebra, Akademie-Verlag, Berlin, 1979.
[20] S.E. Rodabaugh, Axiomatic foundations for uniform operator quasi-uniformities, in: S.E. Rodabaugh, E.P. Klement (Eds.), Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic, vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, pp. 199–233.
[21] S.E. Rodabaugh, E.P. Klement (Eds.), Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic, vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.
[22] J.W. Tukey, Convergence and uniformity in topology, Ann. Math. Studies 1 (1940).
[23] A. Weil, Sur les espaces á structure uniforme et sur la topologie générale, Hermann, Paris, 1937.
[24] D. Zhang, Stratified Hutton uniform spaces, Fuzzy Sets and Systems 131 (2002) 337–346.